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Topology in Entropy of Schwarzschild Black Hole

Guo-Hong Yang[1]

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In the light of ϕ-mapping method and the relationship between the entropy and the Euler characteristic, the inner topological structure of the entropy of Schwarzschild black hole is studied. By introducing an entropy density, it is shown that the entropy of Schwarzschild black hole is determined by the singularities of the timelike Killing vector field of spacetime and these singularities carry the topological numbers, Hopf indices and Brouwer degrees, naturally. Taking account of the statistical meaning of entropy in physics, the entropy of Schwarzschild black hole is merely the sum of the Hopf indices, which will give the increasing law of entropy of black holes.

KEY WORDS: entropy; Euler characteristic; killing vector field.

In recent years, the traditional Bekenstein–Hawking law of entropy of black holes, which is known to be proportional to the area A of event horizon, is argued to be not valid for the extremal black holes (Hawking *et al.*, 1995; Teitelboim, 1995). Based on the study of topological properties, the entropy of four-dimensional extremal Reissner–Nordström black hole is zero regardless of its nonvanishing area of event horizon. Further study (Gibbons and Kallosh, 1995) shows that the source of such a different behavior of entropy between extremal and nonextremal black holes is due to a change in the topological structure. In the extremal case the presence of the event horizon is no longer associated with a nontrivial topology; the Euler characteristic indeed vanishes for extremal black holes, whereas it is different from zero for the nonextremal ones. All these considerations seem to suggest that the extremal black holes should be considered as a rather different object from the nonextremal ones and, particularly, the spacetime topology plays an essential role in the explanation of the entropy of black holes.

In 1997, Liberati and Pollifrone presented a new formulation of the Bekenstein–Hawking law (Liberati and Pollifrone, 1997), which gives the

[1] Department of Physics, Shanghai University, Shanghai 200436, People's Republic of China; e-mail: yguohong@yahoo.com.

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relationship between the entropy S and the Euler characteristic χ,

$$S = \frac{A}{8}\chi. \tag{1}$$

This new formulation is tested to be valid for almost all known gravitational instantons including both the extremal and nonextremal black holes. Using this formula, Su and Wang (Wang *et al.*, 1998; Wang and Su, 1998) presented recently that there may be two kinds of extremal black holes in the nature by taking two limits in different orders. One is the so-called boundary limit suggested by Zaslavskii (1996). The other is the extreme limit $r_+ = r_-$, where r_+ and r_- are the locations of the event horizon and Cauchy horizon, respectively. In this letter, we will study the inner topological structure of the entropy of Schwarzschild black hole on the base of (1). Since the Schwarzschild black hole has only one horizon, there is no extreme limit. We hope that this simplicity can help us to discuss the inner structure clearer and deeper and, more important, can provide for us the necessary information to study the more complicated black holes.

The Schwarzschild black hole has the metric

$$ds^2 = \left(1 - \frac{2M}{r}\right)dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\varphi^2) \tag{2}$$

with the horizon located at $r = 2M$. Using the Gauss–Bonnet–Chern theorem, it has been shown (Liberati and Pollifrone, 1997) that the Euler characteristic of Schwarzschild black hole is

$$\chi = 2, \tag{3}$$

which leads directly to the Bekenstein–Hawking entropy $S = A/4$ by taking account of the relationship (1). The topology (3) gives the global property of Schwarzschild black hole. In order to study the inner topological structure of entropy, let us consider the Euler characteristic in detail.

For a closed N(even)-dimensional Riemannian manifold M^N, the Euler characteristic $\chi(M^N)$ can be expressed as the volume integral of the Gauss–Bonnet–Chern differential N-form Λ:

$$\chi(M^N) = \int_{M^N} \Lambda, \tag{4}$$

$$\Lambda = \frac{(-1)^{N/2}}{2^N \pi^{N/2} (N/2)!} \epsilon_{a_1 a_2 \cdots a_{N-1} a_N} R^{a_1 a_2} \wedge \cdots \wedge R^{a_{N-1} a_N}, \tag{5}$$

where R^{ab} is the curvature 2-form of M^N. On the basis of an instructive idea of working on the sphere bundle $S(M^N)$, Chern (1944, 1945) has shown that the GBC N-form Λ on M^N can be pulled back to $S(M^N)$ as the exterior derivative of a differential $(N-1)$-form Ω:

$$\pi^*\Lambda = d\Omega, \tag{6}$$

where π^* denotes the pullback of the projection $\pi : S(M^N) \to M^N$. Then, using a recursion method and a section $n : M^N \to S(M^N)$, which is a unit tangent vector field over M^N satisfying

$$n^a(x)n^a(x) = 1 \quad a = 1, 2, \ldots, N, \tag{7}$$

Chern (1959) proved that the $(N-1)$-form Ω on $S(M^N)$ can be read as

$$\Omega = \frac{1}{(2\pi)^{N/2}} \sum_{k=0}^{N/2-1} (-1)^k \frac{2^{-k}(N-2k-2)!}{(N-2k-1)!k!} \Theta_k \tag{8}$$

which is called the Chern form with

$$\Theta_k = \epsilon_{a_1 a_2 \cdots a_{N-2k} a_{N-2k+1} a_{N-2k+2} \cdots a_{N-1} a_N} n^{a_1} Dn^{a_2} \wedge \cdots \wedge Dn^{a_{N-2k}} \wedge R^{a_{N-2k+1} a_{N-2k+2}} \wedge \cdots \wedge R^{a_{N-1} a_N}, \tag{9}$$

where Dn^a is the covariant derivative 1-form of $n^a(x)$. It is noted that π^* maps the cohomology of M^N into that of $S(M^N)$, while n^* performs the inverse operation. Thus $n^*\pi^*$ amounts to the identity and the Euler characteristic $\chi(M^N)$ in (4) can be rewritten as

$$\chi(M^N) = \int_{M^N} \Lambda = \int_{M^N} n^*\pi^*\Lambda = \int_{M^N} n^*\, d\Omega. \tag{10}$$

In the opinion of decomposition of gauge potential, Duan *et al.* (1993, 1998) show that the $(N-1)$-form Ω can be formulated in terms of the unit tangent vector field $n^a(x)$ cleanly as

$$\Omega = \frac{1}{(N-1)!A(S^{N-1})} \epsilon_{a_1 a_2 \cdots a_N} n^{a_1}\, dn^{a_2} \wedge \cdots \wedge dn^{a_N}, \tag{11}$$

where $A(S^{N-1})$ is the area of S^{N-1}

$$A(S^{N-1}) = 2\pi^{N/2}/\,\Gamma(N/2). \tag{12}$$

Therefore the Euler characteristic $\chi(M^N)$ is

$$\chi(M^N) = \frac{1}{(N-1)!A(S^{N-1})} \int_{M^N} \epsilon^{\mu_1 \cdots \mu_N} \epsilon_{a_1 \cdots a_N} \partial_{\mu_1} n^{a_1} \cdots \partial_{\mu_N} n^{a_N}\, d^N x. \tag{13}$$

For the four-dimensional Schwarzschild black hole, (13) becomes

$$\chi = \frac{1}{12\pi^2} \int \epsilon^{\mu\nu\lambda\rho} \epsilon_{abcd}\, \partial_\mu n^a \partial_\nu n^b \partial_\lambda n^c \partial_\rho n^d\, d^4 x. \tag{14}$$

For the case considered here, $n^a(x)$ coincides with the direction field of the timelike Killing vector field of spacetime. According to the relationship (1), we introduce the entropy density ρ of Schwarzschild black hole:

$$\rho = \frac{A}{8} \times \frac{1}{12\pi^2} \epsilon^{\mu\nu\lambda\rho} \epsilon_{abcd} \partial_\mu n^a \partial_\nu n^b \partial_\lambda n^c \partial_\rho n^d, \tag{15}$$

where A is the area of horizon. Then the entropy S of black hole is

$$S = \int \rho \, d^4 x. \tag{16}$$

In the following, we will consider the inner topological structure of the entropy through the density ρ and the so-called ϕ-mapping method.

Since $n^a(x)$ is a unit tangent vector field, it can, in general, be further expressed as

$$n^a(x) = \frac{\phi^a(x)}{\|\phi(x)\|}, \quad \|\phi(x)\| = \sqrt{\phi^a(x)\phi^a(x)}, \tag{17}$$

where

$$\phi^a(x) = e^a_\mu(x)\phi^\mu(x), \quad \mu, a = 1, 2, 3, 4 \tag{18}$$

in which $e^a_\mu(x)$ and $\phi^\mu(x)$ are the vielbein and timelike Killing vector field of Schwarzschild black hole, respectively. It is obvious that the singularities of $n^a(x)$ are just the zeros of $\phi^a(x)$. Using

$$\partial_\mu n^a(x) = \frac{1}{\|\phi(x)\|} \partial_\mu \phi^a(x) + \phi^a(x) \partial_\mu \left(\frac{1}{\|\phi(x)\|} \right) \tag{19}$$

and

$$\frac{\partial}{\partial \phi^a} \left(\frac{1}{\|\phi\|} \right) = -\frac{\phi^a}{\|\phi\|^3}, \tag{20}$$

the entropy density ρ in (15) is changed into

$$\rho = -\frac{A}{8} \times \frac{1}{24\pi^2} \epsilon^{\mu\nu\lambda\rho} \epsilon_{abcd} \partial_\mu \phi^i \partial_\nu \phi^b \partial_\lambda \phi^c \partial_\rho \phi^d \frac{\partial}{\partial \phi^i} \frac{\partial}{\partial \phi^a} \left(\frac{1}{\|\phi\|^2} \right). \tag{21}$$

If we define the Jacobian determinant $J(\phi/x)$ as

$$\epsilon^{ibcd} J(\phi/x) = \epsilon^{\mu\nu\lambda\rho} \partial_\mu \phi^i \partial_\nu \phi^b \partial_\lambda \phi^c \partial_\rho \phi^d \tag{22}$$

and make use of the Green's function relation (Gel'fand and Shilov, 1958)

$$\Delta_\phi \left(\frac{1}{\|\phi\|^2} \right) = -4\pi^2 \delta(\phi), \tag{23}$$

where $\Delta_\phi = \partial^2 / \partial\phi^a \partial\phi^a (a = 1, 2, 3, 4)$ is the four-dimensional Laplacian operator in ϕ-space, we do obtain the δ-function like entropy density ρ:

$$\rho = \frac{A}{8} \delta(\phi) J\left(\frac{\phi}{x}\right). \tag{24}$$

It is easy to see that $\rho \neq 0$ only when $\phi(x) = 0$. This result shows that the inner structure of entropy of Schwarzschild black hole is determined by the zeros of $\phi^a(x)$, i.e., the singularities of $n^a(x)$. So we will expand $\delta(\phi)$ in terms of these zeros in the following.

Suppose that the timelike Killing vector field $\phi^a(x)$ possesses l isolated zeros and let the ith zero be

$$x^\mu = z_i^\mu, \qquad \mu = 1, 2, 3, 4, \quad i = 1, 2, \ldots, l. \tag{25}$$

Then, as we proved in Yang *et al.* (2001), the δ-function $\delta(\phi)$ can be expanded by these zeros as

$$\delta(\phi) = \sum_{i=1}^{l} \frac{\beta_i}{|J(\phi/x)_{\mathbf{z}_i}|} \delta(\mathbf{x} - \mathbf{z}_i) \tag{26}$$

where the positive integer β_i is called the Hopf index of ϕ-mapping at $\mathbf{z}_i$ and it means that, when the point $\mathbf{x}$ covers the neighborhood of $\mathbf{z}_i$ once, the function $\phi(x)$ covers the corresponding region β_i times, which is a topological number of the first Chern class and relates to the generalized winding number of ϕ-mapping. Substituting (26) into (24), the inner topological structure of the entropy density ρ is formulated by

$$\rho = \frac{A}{8} \sum_{i=1}^{l} \beta_i \eta_i \delta(\mathbf{x} - \mathbf{z}_i), \tag{27}$$

where

$$\eta_i = \operatorname{sign} J\left(\frac{\phi}{x}\right)\Big|_{\mathbf{z}_i} = \pm 1 \tag{28}$$

is called the Brouwer degree of ϕ-mapping at $\mathbf{z}_i$ (Yang *et al.*, 2001). So, from (16), the entropy S of Schwarzschild black hole is given by the sum of these Hopf indices and Brouwer degrees

$$S = \frac{A}{8} \sum_{i=1}^{l} \beta_i \eta_i, \tag{29}$$

which is the direct result of the combination of the relationship (1) and the Hopf index theorem. However, there is a contradiction in (29). In statistical physics, it is well known that the entropy is determined by the number of the microstates of system. For Schwarzschild black hole, if we have two singularities labeled by the topological numbers $\beta_1 = \beta_2$ but $\eta_1 = -\eta_2$, we have the entropy $S \neq 0$ from the viewpoint of statistics but $S = 0$ from (29). We think that this difference comes essentially from the label of microstates. In statistical physics, all the microstates are marked by the same labels, and the entropy is given by the sum of these labels. But in (29), the singularities are labeled by the Hopf indices and Brouwer degrees, $\beta_i \eta_i$, which can be contrary numbers for two of the singularities. Then, they cancel out each other in the sum and have no contribution to the entropy of system. Now, in order to study the inner topological structure and origin of entropy of Schwarzschild black hole, we take the opinion of statistical physics and so the entropy in (29) is changed into

$$S = \frac{A}{8} \sum_{i=1}^{l} \beta_i, \tag{30}$$

which will be discussed later.

In summary, using the relationship between the entropy and the Euler characteristic, we introduce the entropy density to describe the inner topological structure of the entropy of Schwarzschild black hole. From the ϕ-mapping method, it is shown that the entropy is determined by the singularities of the timelike Killing vector field of spacetime and these singularities are labeled by the topological numbers, Hopf indices and Brouwer degrees, naturally. Taking account of the statistical meaning of entropy, the entropy is given by the Hopf indices merely in (30). Since it is determined by the number of microstates of system, the entropy is essentially a quantum-theoretical quantity. The results (27) and (30) just input the information of quantization through the singularities and the topological numbers, which can be looked upon as the topological quantization of entropy. Meanwhile, with the infall of mass, new singularities might be created in pair or one singular be bifurcated into several singularities. Although the topological quantum numbers, Hopf indices and Brouwer degrees, are conserved during these generation or bifurcation processes, the sum of the Hopf indices will be increased, which will give the increasing law of entropy of Schwarzschild black hole. This idea will be detailed later.

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